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TRANSIENT NATURAL CONVECTION FOR VERTICAL ELEMENTS FOR TIME DEPENDENT INTERNAL ENERGY GENERATION—APPRECIABLE THERMAL CAPACITY

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Abstract—Temperature response has been determined for natural convection transients involving an element of finite thermal capacity (in an extensive fluid) subject to a linear increase in input thermal energy to an asymptotic value. Response characteristics were obtained by numerically integrating the differential equations which resulted from a modified integral analysis of such transients. The calculations indicate the nature of transient response and indicate the conditions under which such transients may be accurately treated as quasi-static processes.

NOMENCLATURE

b, generalizing factor for time;
c, specific heat of fluid;
c'', thermal capacity of element per unit surface area;
g, local gravitational acceleration;
h, local surface coefficient;
k, thermal conductivity;
q, flux time constant, equation (10);
q'', instantaneous energy generation rate per unit of element surface area;
u_g, a velocity, proportional to that achieved in steady state;
u_m, instantaneous local velocity maximum;
x, vertical distance from lower edge of element for heating and from upper edge of element for cooling;
y, distance out, normal to surface;
L, height of element;
M_o, derivative of the generalized temperature distribution, value listed in Table 1;
Gr, conventional Grashof number, based upon L and average temperature excess at infinite time, absolute value;
Gr^*, modified Grashof number, based upon L and surface flux at infinite time, absolute value;
Nu, Nusselt number based upon L and average surface coefficient;
Pr, Prandtl number;
Q, constant related to the element storage capacity, equation (9);
S, Prandtl number dependent constant;
t, temperature;
T, generalized time variable, equation (8);
U, Prandtl number dependent constant;
W, Prandtl number dependent constant;
Y, $\Delta_\theta/\Delta_{\theta,\infty}$.

Greek symbols
α, thermal diffusivity of fluid;
β, coefficient of themal expansion of fluid;
δ_θ, thickness of thermal boundary layer;
Δ_θ, δ_θ/L;
θ, local temperature excess $(t - t_r)$;
θ_m, instantaneous local temperature maximum (or minimum);
μ, fluid aboslute viscosity;
ρ, density of fluid;
τ, time;
ν, fluid kinematic viscosity;
χ, u_m/u_g;
ψ, $\theta_m/\theta_{m,\infty}$;
$\bar{\theta}_m$, $\bar{\psi}$, $\bar{Y}$, and $\bar{\chi}$ are average values over the height of the element.

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Subscripts

- ∞, at infinite time;
- 0, at solid-fluid interface;
- s, quasi-static;
- c, one-dimensional conduction;
- r, in the remote fluid.

INTRODUCTION

In a previous paper [1] the present writer presented an integral method of analysis for transient natural convection from vertical elements immersed in an extensive body of fluid. The various velocity and temperature distribution constants which arose in the analysis were evaluated for various Prandtl numbers from 0·01 to 1000 from the vertical flat plate solutions of Ostrach [2]. Therefore, the equations, with these constants, apply for vertical elements of relatively large radius of curvature. In [1] solutions of the general equations were obtained for a step in thermal flux at the surface of an element having zero thermal capacity. The element temperature response was essentially independent of the Prandtl number in the generalized variables used. The response was almost the same as a one-dimensional conduction transient for the first 80 per cent of the temperature rise.

A subsequent paper [3] presents solutions of the general equations for a step in thermal energy input to elements having appreciable thermal capacity. Again the Prandtl number effect is negligible in terms of the generalized variables. Three types of transients were found, depending upon the value of a generalized thermal capacity parameter Q. For $Q < 0{\cdot}1$, processes are essentially one-dimensional conduction in the fluid (allowing for element thermal capacity). For $Q > 1{\cdot}0$, processes are essentially quasi-static, the transport process being essentially equivalent at all times to a steady state process at the instantaneous temperature condition. For Q between 0·1 and 1·0 the processes are true convection transients in the sense that the fluid thermal capacity and inertia effects are important. The initial rate of temperature response is Q^{-1} in all cases.

In a companion paper [4] the results of the analysis in [3] are compared with measured transients on vertical plates in air and in water. Temperature measurements in the tests in air were made with an optically chopped, lead-selenide, infrared detector having a time constant of approximately 30 μs. Various test data in water, available in the literature in [5], [6] and [7], are based upon resistance thermometry and upon interferograms. The air tests were predicted to be essentially quasi-static and the water tests were predicted to be essentially one-dimensional conduction. All of the test data are in complete agreement with the results of the theory. This may be seen in Fig. 3 by comparing the experimental points with the solid line curves.

Treatments of natural convection transients by other workers are summarized in [1] and in [3]. These analyses treat a variety of cases and indicate the present limitations of exact and perturbation solutions. None of these analyses include the effect of element thermal capacity, which is inevitably present in actual equipment.

In the present paper the effect upon temperature response of a time dependent thermal energy input to the element is investigated. Since the necessary condition for an essentially quasi-static response is not particularly stringent even for a step in input, a less extreme variation, a linear increase in input to the asymptotic value, is investigated. Such a linear increase is a good approximation for some reactor transients and for many transients in electrical equipment.

SOLUTIONS FOR TRANSIENTS

The analysis of [1] resulted in ordinary differential equations which relate the instantaneous value of the element temperature, the thermal layer thickness and the maximum induced velocity variables (ψ, Y, and χ) to generalized time T. The equations apply to vertical plates and to vertical cylinders, sketched in Fig. 1, under the conditions in which a two-dimensional laminar boundary layer analysis in Cartesian co-ordinates is permissible. The equations in terms of average values ($\bar{\psi}$, $\bar{Y}$, and $\bar{\chi}$), averaged over the height of the element at time T, are written as

$$\frac{\bar{\psi}}{\bar{Y}} - a\frac{\mathrm{d}}{\mathrm{d}T}(\bar{\psi}\bar{Y}) - \bar{Y}\bar{\chi}\bar{\psi} = 0 \qquad (1)$$

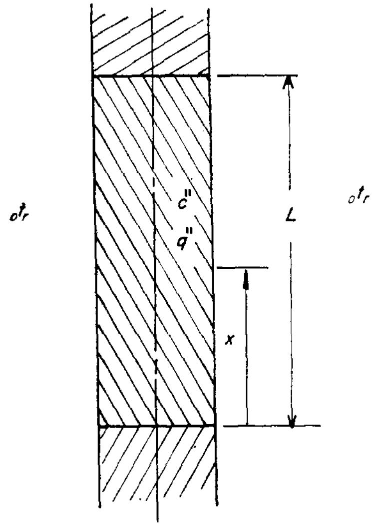

FIG. 1.

$$S\bar{\psi}\bar{Y} - U\frac{\bar{\chi}}{\bar{Y}} - \frac{d}{dT}(\bar{\chi}\bar{Y}) - W\bar{Y}\bar{\chi}^2 = 0 \quad (2)$$

$$\frac{\bar{\psi}}{\bar{Y}} = \frac{q''}{q''_\infty} - Q\frac{d\bar{\psi}}{dT} \quad (3)$$

where the constants S, U, W, and a depend only upon Prandtl number. Note that in steady state the time derivatives are zero and that $\bar{\psi}$, $\bar{Y}$, and $\bar{\chi}$ are 1·0. Therefore, from equation (2) we have

$$S = U + W. \quad (4)$$

The thermal flux quantities, q'' and q''_∞, are the instantaneous and asymptotic values of the rate of energy input to the element per unit of element surface area. The dependent variables $\bar{\psi}$, $\bar{Y}$, and $\bar{\chi}$, the generalized time T, and the generalized thermal capacity variable Q are defined as follows:

$$\bar{\psi} = \frac{\bar{\theta}_m}{\bar{\theta}_{m,\infty}} \quad (5)$$

$$\bar{Y} = \frac{\bar{\Delta}_\theta}{\bar{\Delta}_{\theta,\infty}} \quad (6)$$

$$\bar{\chi} = u_m/u_g \quad (7)$$

$$T = \frac{\alpha\tau}{L^2\Delta^2_{\theta,\infty}} = \frac{\alpha\tau}{L^2}(bGr^*Pr)^{2/5} \quad (8)$$

$$Q = \frac{c''}{\rho cLM_o}b(Gr^*Pr)^{1/5}. \quad (9)$$

The Prandtl number dependent quantities are listed in Table 1 and the various quantities are defined in the listing of notation.

The linear increase in input flux to the element, which is the case considered here, is written in terms of the generalized time T as follows:

$$\frac{q''}{q''_\infty} = qT \quad (10)$$

where q is the time constant of the increase. The flux ratio from this relation is used until $qT = 1{\cdot}0$. Thereafter the ratio is taken as 1·0. This variation is shown in Fig. 2 and is an idealization of an increase in input flux to an asymptotic value.

Equations (1), (2), (3) and (10) may be combined to eliminate $\bar{Y}$ and q''/q''_∞ to give the following differential equations for $\bar{\psi}$ and $\bar{\chi}$, valid during the increase.

Table 1. Values of the Prandtl number dependent constants based upon the steady state distributions of Ostrach [2]

Pr	a	S	U	W	M_o	$b \times 10^4$
0·01	0·1844	9·083	1·165	7·918	1·88	1·408
0·72	0·2000	16·13	9·242	6·886	1·79	40·25
1·0	0·1971	17·67	10·97	6·704	1·79	71·15
5†	0·1936	33·40	26·78	6·616	1·78	75·97
10	0·1894	41·69	35·29	6·398	1·77	87·43
100	0·1924	126·9	121·4	5·523	1·77	137·1
1000	0·1905	263·2	258·6	4·677	1·76	118·4

† Interpolated from other values.

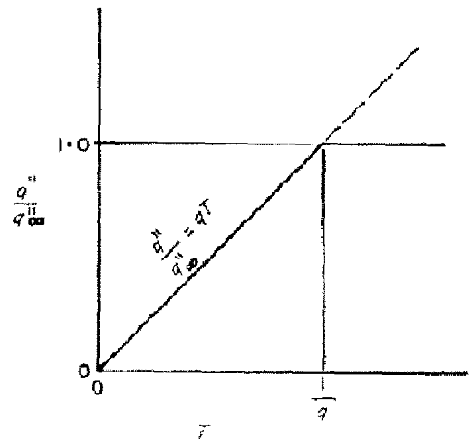

FIG. 2.

$$\bar{\psi}''' = \frac{(qT - Q\bar{\psi}')}{a\bar{\psi}^2 Q}\left[(qT - Q\bar{\psi}')^2 - 2a\bar{\psi}\bar{\psi}' + \frac{aq\bar{\psi}^2}{(qT - Q\bar{\psi}')} - \bar{\chi}\bar{\psi}^2\right] \quad (11)$$

$$\bar{\chi}' = S\bar{\psi} - \frac{U\bar{\chi}}{\bar{\psi}^2}(qT - Q\bar{\psi}')^2 - \frac{X\bar{\psi}'}{\bar{\psi}} - W\bar{\chi}^2 + \frac{q\bar{\chi}}{(qT - Q\bar{\psi}')} - \frac{\bar{\chi}Q\bar{\psi}''}{(qT - Q\bar{\psi}')}. \quad (12)$$

The terms which arise due to a time dependent flux are the third in equation (11) and the fifth in equation (12). For the time period after the linear increase these terms are dropped and the qT is taken equal to 1·0. As in the case of a step in input flux [3], an order of magnitude analysis indicates that $\bar{\psi}/\bar{Y}$ is zero at $T = 0$. Therefore, from equation (3), $\bar{\psi}' = 0$ at $T = 0$ and the initial slope of the temperature response is zero. The boundary conditions are:

$$\text{at } T = 0, \ \bar{\psi} = \bar{\psi}' = \bar{\chi} = 0. \quad (13)$$

The differential equation for the quasi-static response $\bar{\psi}_s$ to a time dependent input may be obtained from equations (1), (2) and (3) by omitting the terms for the acceleration of the fluid and for the time rate of change of thermal energy storage in the fluid.

$$Q\frac{d\bar{\psi}_s}{dT} + \bar{\psi}_s^{5/4} - \frac{q''}{q''_\infty} = 0. \quad (14)$$

This is of the form of a Bernoulli differential equation. Substituting the linear flux variation, equation (10), we have

$$\frac{d\bar{\psi}_s}{d(T/Q)} + \bar{\psi}_s^{5/4} - qQ\left(\frac{T}{Q}\right) = 0 \quad (15)$$

where $\bar{\psi}_s = 0$ when $T = 0$ and the last term becomes 1·0 at $qT = 1·0$. Equation (15) indicates that $\bar{\psi}$ is a function only of T/Q and qQ.

Calculations were carried out for a Prandtl number of 0·72 to determine the conditions under which the response $\bar{\psi}$ was within a few per cent of the quasi-static $\bar{\psi}_s$. Consideration was limited to values of Q less than 1·0 since even for a step in input flux (essentially $q \to \infty$) all processes for which $Q > 1·0$ are approximately quasi-static. Values of Q of 0·5, 0·1, 0·01, and 0·001 were considered.

The calculated temperature response for a linear increase in input flux is compared to a step in input (to the same asymptotic value) in Fig. 3. The solid curves, from [3], apply for a step for various values of the thermal capacity parameter Q. The response for $Q = 0·5$ was calculated as a convection transient. For $Q = 1$ and 10 the responses are essentially the same as the quasi-static response. For $Q = 0·029$, 0·0087 and 0·0014 the responses shown are essentially equivalent to a one-dimensional conduction transient. The experimental points are from tests [4] with a step in input at the values of Q shown on the figure. This plot shows the excellent agreement between the theory and measurements for the case of a step in input.

The dashed curves on Fig. 3 are convection transients calculated for linear increases at the rates $q = 1$ and $q = 5$ for $Q = 0·5$. The two dashed curves show that the initial nature of the response for a linear increase is quite different, even for the very short "rise-times" associated with values of q of 1 and 5.

The difference in response for the two types of input flux variation may be seen in more detail in Fig. 4 where convection transients for $Q = 0·1$ are compared. The upper solid line curve is for a step in input, the limiting rate for the linear increase, i.e. $q = \infty$, and the others are for various values of q. The quasi-static solutions, from [3] and from equation (15), are shown as

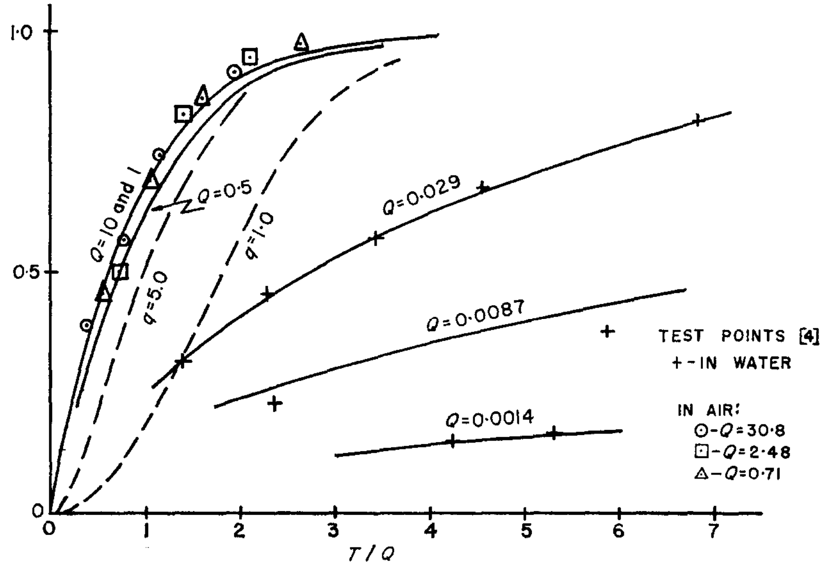

FIG. 3. Temperature responses for various flux input conditions, comparisons with measurements.

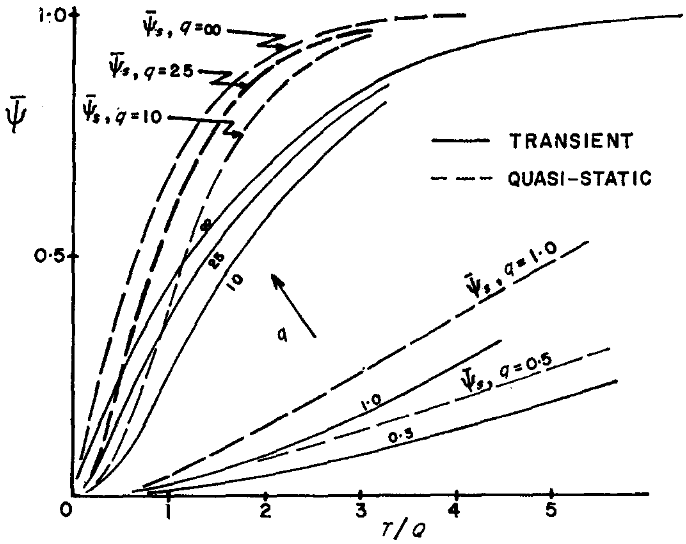

FIG. 4. Temperature responses (for $Q = 0{\cdot}1$).

dashed curves. Values of q are indicated by the numbers on the curves. There is seen to be a substantial transient effect for the more rapid rates of input flux increase, i.e. for larger values of q.

For the step increase in flux input, the condition for an essentially quasi-static process was not particularly extreme. The condition is merely $Q > 1{\cdot}0$. For the case of a linear increase it seemed likely that, for sufficiently slow increases in input, i.e. small q, quasi-static processes should be found even for $Q < 1{\cdot}0$. This possibility was investigated by calculating $\bar{\psi}$ and $\bar{\psi}_s$ from equations (11), (12) and (15) for $Q = 0{\cdot}5$, 0·1, 0·01 and 0·001 for various q. For each value of Q the value of q was determined for which $\bar{\psi}_s - \bar{\psi}$ remained less than 0·05 over the entire range of $\bar{\psi}$ from 0 to 1·0. That is, the value of q was determined for which $\bar{\psi}_s$ gave the correct temperature response within 5 per cent of the asymptotic value of $\bar{\psi}$. For smaller values of q than this limiting one, the quasi-static response is an even better approximation. These limiting values of q are plotted in Fig. 5 against Q and a curve is drawn through the points, thereby separating the regimes of essentially quasi-static and true transient. Note that from the results in [3] the quasi-static is within 5 per cent of the convection transient for $Q = 1{\cdot}0$ even for a step in flux input. It is to be noted that the maximum of $\bar{\psi}_s - \bar{\psi}$ occurs very early in the transient. Therefore, if the general temperature response (or the time to essentially steady state) are the matters of primary interest, the simple quasi-static result may be used for much larger values of q than those suggested by Fig. 5.

CONCLUSIONS

Natural convection transients have been analysed by numerically integrating equations derived through an extension of the integral method of boundary layer analysis. Various types of processes have been treated. Consideration of processes resulting from a step in thermal energy input to an element having finite thermal capacity have indicated the Prandtl number effect and have delineated three types of responses: the essentially quasi-static, the true convection transient, and the essentially one-dimensional conduction response.

The results reported in this paper indicate the nature of transient response for a linear increase

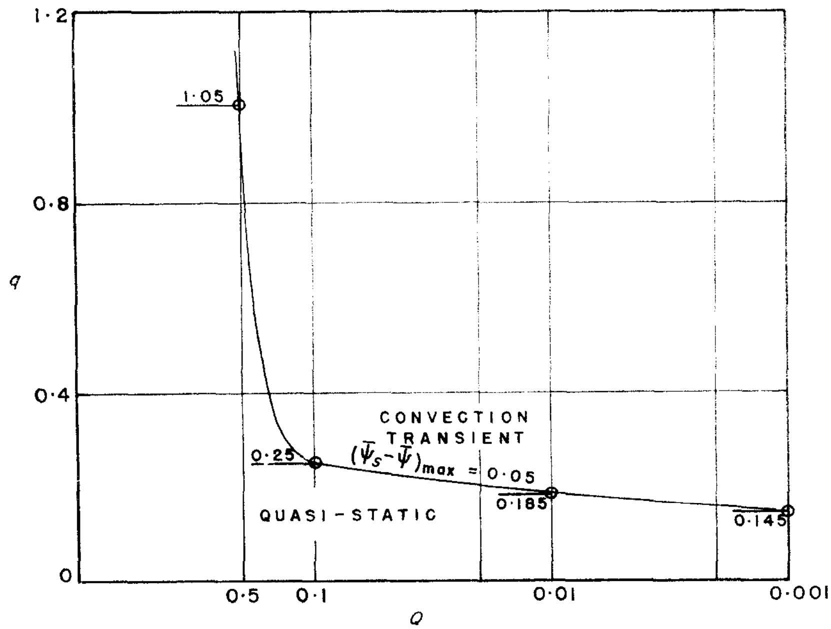

FIG. 5. Response regimes for a linear increase in input flux.

in thermal energy input and show that, even for elements of relatively low thermal capacity, a quasi-static process may result for sufficiently low rates of flux increase. The limits for such a response are given for a broad range of element thermal capacity.

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Résumé—Les réponse thermiques ont été déterminées pour des cas de convection naturelle transitoire sur un élément de capacité thermique finie placé dans un fluide s'étendant à l'infini, soumis à une loi de chauffage linéaire jusqu'au régime asymptotique. Les réponses sont obtenues par intégration numérique des équations différentielles résultant d'une analyse intégrale des réponses transitoires. Les calculs donnent la nature des réponses transitoires et indiquent les conditions sous lesquelles de tels problèmes peuvent être traités avec précision en temps que processus quasi-statique.

Zusammenfassung—Für Anlaufvorgänge bei freier Konvektion an einem Element endlicher Wärmekapazität (in einem ausgedehnten Medium), das einer linearen Heizleistungssteigerung bis zu einem asymptotischen Wert unterworfen war, wurde der Temperaturverlauf bestimmt. Die Charakteristik dieses Temperaturverlaufs liess sich durch numerische Integration der Differentialgleichungen, die aus einer modifizierten Integralanalyse der Anlaufvorgänge stammen, erhalten. Die Rechnungen zeigen die Art des Anlaufvorganges und geben die Bedingungen an, für die derartige instationäre Vorgänge als quasi-stationäre Probleme genau behandelt werden können.

Аннотация—Определено изменение температуры при неустановившихся процессах естественной конвекции для элементов конечной теплоемкости (в большом объеме жидкости) с линейным увеличением подачи тепловой энергии до асимптотического значения. Получены характеристики переходного процесса путем численного интегрирования дифференциальных уравнений, выведенных на основе модифицированного интегрального анализа таких неустановившхся процессов. Расчеты указывают на природу характеристики неустановившегося режима и те условия, при которых такие неустановившиеся режимы могут рассматриваться как квазистатические процессы.